TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 309, Number 2, October 1988

ON THE NONLINEAR EIGENVALUE PROBLEM $\Delta u + \lambda e^u = 0$

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ABSTRACT. The structure of the set $\mathscr{C}$ of solutions of the nonlinear eigenvalue problem $\Delta u + \lambda e^u = 0$ under Dirichlet condition in a simply connected bounded domain Ω is studied. Through the idea of parametrizing the solutions (u, λ) in terms of $s = \lambda \int_\Omega e^u\, dx$, some profile of $\mathscr{C}$ is illustrated when Ω is star-shaped. Finally, the connectivity of the branch of Weston-Moseley's large solutions to that of minimal ones is discussed.

1. Introduction. Our purpose is to study the nonlinear eigenvalue problem (P):

$$-\Delta u = \lambda e^u \quad (\text{in } \Omega) \tag{1.1}$$

under the Dirichlet boundary condition

$$u = 0 \quad (\text{on } \partial\Omega), \tag{1.2}$$

where $\Omega \subset \mathbf{R}^2$ is a simply connected and bounded domain with smooth boundary $\partial\Omega$ and when $\lambda > 0$. We are seeking the solution $h = {}^T(u, \lambda)$ of (P) which is taken in the classical sense so that $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. If we fix λ and regard (P) just as a nonlinear elliptic equation, then its solution u is called a section at λ of the original eigenvalue problem.

Our problem arises in differential geometry and also in mathematical physics and has been studied by several authors [**6**, **12**, **5**, **13**, **9**, **19**, **11**, **1**, **2**, **4**]. From these works we know the following, where "branch" means a portion of a one-dimensional manifold in $\mathbf{R} \times C^0(\bar{\Omega})$:

(i) There is a branch $\mathscr{C}_0$ of solutions $(\lambda, u) = (\lambda_t, u_t)$ $(0 \le t < 1)$ for (P), which originates from $(\lambda, u) = (0, 0)$ at $t = 0$ and goes toward $\lambda > 0$ as $t > 0$.

(ii) That branch $\mathscr{C}_0$, without any bifurcation, continues up to $\lambda = \bar{\lambda}$ for some $\bar{\lambda} = \bar{\lambda}(\Omega)$ in $0 < \bar{\lambda} < \infty$ and then turns to $\lambda < \bar{\lambda}$, that is, the bending occurs. In other words, in the parametrization $\mathscr{C}_0 = \{(\lambda_t, u_t) | 0 \le t < 1\}$, there exists a $\bar{t}$ in $(0, 1)$ such that $\lambda_t \uparrow \bar{\lambda}$ as $(t \uparrow \bar{t})$ and $\lambda_t \downarrow$ for $\bar{t} < t < 1$. Furthermore, the component of the solutions for (P) containing $\mathscr{C}_0$ is unbounded.

We set $\underline{\mathscr{C}} = \{(\lambda_t, u_t) | 0 \le t \le \bar{t}\} \subset \mathscr{C}_0$.

(iii) The branch $\underline{\mathscr{C}}$ is minimal in the sense that for any section $u = u(x)$ at $\lambda = \lambda_t$ $(0 \le t \le \bar{t})$, the relation $u_t(x) \le u(x)$ $(x \in \Omega)$ follows. Furthermore, here the equality holds at some $x \in \Omega$ if and only if $u \equiv u_t$.

(iv) When $\lambda > \bar{\lambda}$, there is no section u of (P). On the other hand, for $0 < \lambda < \bar{\lambda}$ there is a section u such that $(u, \lambda) \notin \underline{\mathscr{C}}$. Therefore, at least two sections exist at each λ in $0 < \lambda < \bar{\lambda}$.

Received by the editors May 2, 1986 and, in revised form, July 4, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35P30; Secondary 35J60.

Recently, under certain assumptions for Ω, V. H. Weston and J. L. Moseley have constructed a branch $\mathscr{C}^*$ differing from $\underline{\mathscr{C}}$ by the method of singular perturbations [**22**, **16**]. In the parametrization $\mathscr{C}^* = \{(\lambda_t, u_t) | 2 < t < 3\}$, we have

$$\lambda_t \downarrow 0 \quad \text{and} \quad u_t(x) \to 4 \log |1 - \bar{\delta} g^{-1}(z)| / |g^{-1}(z) - \delta|$$

as $t \uparrow 3$, where $z = x_1 + ix_2 \in \mathbf{C}$ for $x = (x_1, x_2) \in \mathbf{R}^2$. Here, $g \colon D = \{|\varsigma| < 1\} \to \Omega$ is a Riemann mapping, that is, one-to-one and conformal mapping having a diffeomorphic extension $\bar{g} \colon \bar{D} \to \bar{\Omega}$. Furthermore, $\delta \in D$ solves the equation

$$\bar{\delta} = \tfrac{1}{2}(1 - |\delta|^2) g''(\delta) / g'(\delta). \tag{1.3}$$

Henceforth, $\mathscr{C}^*$ is called the branch of Weston-Moseley's large solutions.

The main object of the present paper is to show that if Ω is close to a disc, then $\underline{\mathscr{C}}$ and $\mathscr{C}^*$ are connected to each other and form one branch of solutions, which may be denoted by $\mathscr{C} = \{(\lambda_t, u_t) | 0 \leq t < 3\}$.

We note that the branch of large solutions actually connects with that of minimal solutions, in the case $\Omega = D \equiv \{|\varsigma| < 1\}$. In fact, $f(u) = \lambda e^u > 0$ and hence $u > 0$ in Ω. Therefore, by a theorem due to Gidas, Ni, and Nirenberg [**7**], every section $u = u(x)$ of (P) is radially symmetric: $u = u(|x|)$. Consequently, from the results of Gel'fand [**6**] we have $\bar{\lambda}(D) = 2$ and that (P) for $\Omega = D$ has exactly two sections at λ in $0 < \lambda < 2$. Actually, these are given as

$$\left(\frac{\lambda}{8}\right)^{1/2} e^{u/2} = \frac{\rho^{1/2}}{|x|^2 + \rho} \quad \text{with } \rho^{1/2} = \rho_{\pm}^{1/2} = \left(\frac{\lambda}{2}\right)^{-1/2} \left\{1 \mp \sqrt{1 - \frac{\lambda}{2}}\right\}. \tag{1.4}$$

2. Preliminaries. 1. We first look at Weston-Moseley's theory briefly and afterwards give some remarks.

They make use of the Liouville integral [**14**] for the equation (1.1) to construct asymptotic solutions $u = u^n$ $(n = 1, 2, \ldots)$ for (P) as $\lambda \downarrow 0$ under a certain assumption, which we shall describe later. Namely, $u = u^n$ satisfies (1.1) with

$$u^n = O(\lambda^n) \quad (\text{on } \partial\Omega) \text{ as } \lambda \downarrow 0, \tag{1.2'}$$

and is given explicitly in terms of the Riemann mapping $g \colon D \to \Omega$. In fact, it behaves like

$$u^n(x) \sim 4 \log |1 - \bar{\delta} g^{-1}(z)| / |g^{-1}(z) - \delta|$$

as $\lambda \downarrow 0$, where $\delta \in D$ solves the equation (1.3).

It holds that the solution $\delta \in D$ of (1.3) is characterized as $\delta = g^{-1}(d)$, where $d \in \Omega$ is a point of maximal conformal radius for Ω [**16**, p. 721]. Therefore, such a $\delta \in D$ exists for each simply connected domain $\Omega \subset \mathbf{R}^2$. Further, d is unique when Ω is convex. (See [**16**, **8**] and also [**20**, **10**].) Now, construct another Riemann mapping $g_N \colon D \to \Omega$ just by composing $\varphi(\varsigma) = (\varsigma - \delta)/(1 - \bar{\delta}\varsigma)$ to g from the right-hand side. Then, $\delta \in D$ can be reduced to $0 \in D$, and (1.3) is equivalent to

$$g_N''(0) = 0. \tag{2.1}$$

In this notation, a simple sufficient condition for the existence of the asymptotic solutions described above has been given by Moseley [**16**]. That is,

$$\alpha = \alpha(d, \Omega) = |g_N''(0)/g_N'(0)| \neq 2. \tag{2.2}$$

Moseley [**16**] further showed $\alpha < 2$ in the case that Ω is convex.

Genuine solutions for (P) are constructed by a Newton-like iteration. Namely, first we pull back the problem (P) in Ω to that in D by $g_N: D \to \Omega$:

$$-\Delta U = \lambda |g_N'|^2 e^U \quad (\text{in } D) \tag{2.3}$$

with

$$U = 0 \quad (\text{on } \partial D). \tag{2.4}$$

Through the Green's function

$$K(x, y) = \frac{1}{2\pi} \log \left| \frac{w - z}{1 - \bar{z}w} \right|,$$

where $z = x_1 + ix_2$ and $w = y_1 + iy_2$ for $x = (x_1, x_2)$ and $y = (y_1, y_2)$, respectively, the above problem is transformed into the integral equation

$$U = K(U) \equiv \lambda \int_D K(x, y)(|g_N'|^2 e^U)(y)\, dy. \tag{2.5}$$

Here, the modified-Newton iteration

$$U_{k+1} = S(U_k) \qquad (k = 0, 1, 2, \dots) \tag{2.6}$$

is applied where $S(U) = (1 - K'_{U_0})^{-1}(K(U) - K'_{U_0}(U))$. In the case that the iteration (2.6) converges in $C^0(\bar{D})$, a genuine solution U^* of (2.3) with (2.4) is obtained. It can be shown that if the starting point U_0 satisfies

$$||U_0 - K(U_0)||_{C^0(\bar{D})} \leq \log((1 + \Gamma)/\Gamma) - (1 + \Gamma)^{-1},$$

then (2.6) converges, where Γ is a positive constant such that

$$||(1 - K'_{U_0})^{-1} K'_{U_0}|| \leq \Gamma.$$

Furthermore, we have

$$||U^* - U_0||_{C^0(\bar{D})} \leq \log((1 + \Gamma)/\Gamma). \tag{2.7}$$

See Weston [**22**, p. 1040].

When the nth asymptotic solution $U^n = u^n \circ g_N$ is taken as a starting point U_0 in the scheme (2.6), we have

$$||U_0 - K(U_0)||_{C^0(\bar{D})} \leq C\lambda^n \quad \text{as } \lambda \downarrow 0$$

with a constant $C > 0$ from (1.1) with (1.2′). On the other hand, by the method of Weston [**22**], we get

$$||(1 - K'_{U_0})^{-1}||_{C^0(\bar{D}) \to C^0(\bar{D})} \leq C\lambda^{-1} \quad \text{as } \lambda \downarrow 0$$

except for a "pathological case" of Ω. Therefore, for $n \geq 3$ the iteration (2.6) converges to a genuine solution U^* such that

$$||U^* - U_0||_{C^0(\bar{D})} \leq C\lambda^{(n-1)/2} \quad \text{as } \lambda \downarrow 0, \tag{2.8}$$

provided that $\lambda > 0$ is small. In fact, we can take $\Gamma = C_1 \lambda^{-1-l}$ $(l \geq 0)$ by

$$||(1 - K'_{U_0})^{-1} K'_{U_0}|| \leq 1 + ||(1 - K'_{U_0})^{-1}|| \leq C_2 \lambda^{-1} \qquad (\lambda \downarrow 0).$$

Then,

$$||U_0 - K(U_0)||_{C^0(\bar{D})} \leq \log((1 + \Gamma)/\Gamma) - (1 + \Gamma)^{-1}$$

holds if $n = 2l + 3$ and $\lambda \downarrow 0$. Further, then

$$||U^* - U_0||_{C^0(\bar{D})} \le \log((1+\Gamma)/\Gamma) \le C_3\lambda^{1+l} = C_3\lambda^{(n-1)/2}.$$

Here, C_1, C_2 and C_3 are positive constants.

By the method of Wente [**21**], it can be shown that the "pathological case" does not arise when $\alpha \equiv |g_N''(0)/g_N'(0)| < 2$. The function $u^* = u_\lambda^* = U^* \circ g_N^{-1}$ becomes a nonminimal section for (P), and the branch $\mathscr{C}^* = \{(\lambda, u_\lambda^*)\}$ of large solutions has been constructed.

From the inequality (2.8) and the concrete expression of U_0 $(= u_n \circ g_N)$, we can derive an important relation,

$$S \equiv \lambda \int_\Omega e^{u_\lambda^*}\, dx = 8\pi + C\lambda + o(\lambda) \quad \text{as } \lambda \downarrow 0, \tag{2.9}$$

with a constant $C = C(d, \Omega)$ defined by

$$\frac{C}{\pi} = -|a_1|^2 + \sum_{n=3}^{\infty} \frac{n^2}{n-2}|a_n|^2, \tag{2.10}$$

where $g_N(\varsigma) = \sum_{n=0}^{\infty} a_n\varsigma^n$ $(a_2 = 0)$. By virtue of Bieberbach's area theorem [**18**, p. 210], we can show the following fact, where $\kappa = \kappa(\varsigma)$ denotes the curvature of $\partial\Omega$ at the point $g_N(\varsigma) \in \partial\Omega$ for $\varsigma \in \partial D$.

PROPOSITION 1. *If $\kappa|g_N'| < 2$ holds everywhere on ∂D, then $C < 0$ follows.* □

In the case that Ω is a disc: $\Omega = \{|z| < R\}$, we have $\kappa|g_N'| \equiv 1$. Further, we note that $C = C(d, \Omega) < 0$ implies that $\alpha = \alpha(d, \Omega)$ $(= 6|a_3/a_1|) < 2$.

The proof of (2.9) with (2.10) and Proposition 1 will be given in Appendices 1 and 2, respectively.

2. We next look over Bandle's theory [**3**] about a priori estimates for solutions and eigenvalues.

Namely, let $h = {}^T(u, \lambda)$ solve (1.1) with $p = \lambda e^u$ (> 0). We consider a surface $\mathscr{M} \equiv (\Omega, d\sigma)$ with the metric $d\sigma^2 = p\,ds^2$ $(= p(dx_1^2 + dx_2^2))$. Then, the surface element and the Gaussian curvature are $d\tau = p\,dx$ $(= p\,dx_1\,dx_2)$ and $K = -(\Delta \log p)/2p = 1/2$, respectively. Bol's inequality is expressed as

$$l(\omega)^2 \ge \tfrac{1}{2}(8\pi - m(\omega))m(\omega) \tag{2.11}$$

for $\omega \subset \Omega$, where $l(\omega) = \int_{\partial\omega} d\sigma$ and $m(\omega) = \int_\omega d\tau$. In the manner of (2.11), we can give the following estimate [**17**].

PROPOSITION 2. *Let $h = {}^T(u, \lambda)$ solve* (P) *and put $S = \lambda \int_\Omega e^u\, dx$. Then, we have*

$$||u||_{C^0(\bar{\Omega})} \le -2\log(1 - S/8\pi), \tag{2.12}$$

provided that $S < 8\pi$. □

Note that as for λ we have $|\Omega|^{-1}e^{-||u||_{C^0(\bar{\Omega})}}S \le \lambda \le \bar{\lambda}$. Estimate (2.12) is also seen in Bandle [**3**, p. 85, Problem]. (Namely, we have only to take $p = \lambda e^u$, $K_0 = 1/2$ and $M = S$, there.) We shall give the third proof in Appendix 3.

We obtain the operator $A_p = -\Delta - p$ under the Dirichlet condition for $p = \lambda e^u$ by linearizing problem (P) with respect to u at the solution $h = {}^T(u, \lambda)$. Let

$\sigma(A_p) = \{\mu_j(p)\}_{j=1}^{\infty}$ $(-\infty < \mu_1(p) < \mu_2(p) \leq \cdots \to +\infty)$ denote its eigenvalues. Then, the relation $\mu_1(p) \geq 0$ holds when $h = {}^T(u, \lambda)$ is minimal, while conversely, $\mu_1(p) > 0$ implies the minimality of h [**4**].

The following proposition is obtained through Schwarz' symmetrization associated with the surface $\mathscr{M}$ [**3**, p. 108]. See also [**17**].

PROPOSITION 3. *For the solution $h = {}^T(u, \lambda)$ of* (P), *the inequality $S \equiv \lambda \int_\Omega e^u\,dx < 4\pi$ implies $\mu_1(p) > 0$.* □

An immediate consequence is the following.

COROLLARY 1. *Similarly, $S < 8\pi$ implies $\mu_2(p) > 0$.*

PROOF. The eigenfunction φ_2 of A_p corresponding to $\mu_2(p)$ has two nodal domains Ω_1 and Ω_2. From the assumption, either $S_1 \equiv \lambda \int_{\Omega_1} e^u\,dx < 4\pi$ or $S_2 \equiv \lambda \int_{\Omega_2} e^u\,dx < 4\pi$ holds. On the other hand, $\mu_2(p)$ may be regarded as the first eigenvalue of the operator $-\Delta - p$ under the Dirichlet condition on Ω_1 or Ω_2. Hence $\mu_2(p) > 0$ follows. □

3. Now, we shall describe our key idea, that is, parametrizing the solution $h = {}^T(u, \lambda)$ of (P) in terms of $S = \lambda \int_\Omega e^u\,dx$ rather than λ. See Nagasaki and Suzuki [**17**] for the background of this idea.

For α in $0 < \alpha < 1$, we set $X = C_0^{2+\alpha}(\bar{\Omega}) \equiv \{v \in C^{2+\alpha}(\bar{\Omega}) | v = 0 \text{ on } \partial\Omega\}$, $Y = C^\alpha(\bar{\Omega})$, $\hat{X} = \underset{\mathbf{R}}{\overset{X}{\times}}$, $\hat{X}_+ = \underset{\mathbf{R}_+}{\overset{X}{\times}}$, and $\hat{Y} = \underset{\mathbf{R}}{\overset{Y}{\times}}$, and define a mapping $\Phi = \Phi(h, S) \colon \hat{X}_+ \times \mathbf{R} \to \hat{Y}$ as

$$\Phi(h, S) = \begin{pmatrix} \Delta u + \lambda e^u \\ \int_\Omega e^u\,dx - \frac{S}{\lambda} \end{pmatrix}$$

for $h = {}^T(u, \lambda)$. Zeros of Φ characterize the solutions $h = {}^T(u, \lambda)$ of (P) such that $S = \lambda \int_\Omega e^u\,dx$. The Fréchet derivative $d_h\Phi \colon \hat{X} \to \hat{Y}$ of Φ with respect to h at (h, S) is given by the matrix

$$d_h\Phi = \begin{pmatrix} \Delta + \lambda e^u & e^u \\ \int_\Omega e^u \cdot\,dx & \frac{S}{\lambda^2} \end{pmatrix}.$$

For the moment, let $(h, S) \in \hat{X}_+ \times \mathbf{R}$ $(h = {}^T(u, \lambda))$ be a zero point of Φ and set $p = \lambda e^u$.

LEMMA 1. *The operator $d_h\Phi \colon \hat{X} \to \hat{Y}$ is invertible if $\mu_1(p) \geq 0$.* □

PROOF. The operator

$$T = d_h\Phi = \begin{pmatrix} -A_p & e^u \\ \int_\Omega e^u \cdot\,dx & \frac{S}{\lambda^2} \end{pmatrix} : \hat{X} \to \hat{Y}$$

has a selfadjoint extension $\tilde{T}$ in $\underset{\mathbf{R}}{\overset{L^2(\Omega)}{\times}}$ with the domain

$$D(\tilde{T}) = \underset{\mathbf{R}}{\overset{H_0^1(\Omega) \cap H^2(\Omega)}{\times}}.$$

Therefore, $\tilde{T}$ is invertible if and only if $\operatorname{Ker} \tilde{T} = \{0\}$, but the same is true for $T = d_h\Phi \colon \hat{X} \to \hat{Y}$ by virtue of the elliptic regularity property of A_p.

Hence, suppose that $f = {}^T(v,\rho) \in \hat{X}$ with $(v,\rho) \neq (0,0)$ is in the kernel of $T = d_h\Phi$. This means that

$$\Delta v + \lambda e^u v + \rho e^u = 0 \quad (\text{in } \Omega), \qquad v = 0 \quad (\text{on } \partial\Omega) \tag{2.13}$$

and

$$\int_\Omega e^u v\,dx + \frac{\rho S}{\lambda^2} = 0. \tag{2.14}$$

Multiply (2.13) by v and integrate

$$T(v) \equiv \int_\Omega |\nabla v|^2\,dx - \lambda\int_\Omega e^u v^2\,dx = \rho\int_\Omega e^u v\,dx = -\frac{\rho^2 S}{\lambda^2}.$$

If $\rho \neq 0$, then $T(v) < 0$ which implies that $\mu_1(p) < 0$. If $\rho = 0$, then $v =$ constant $\times\varphi_1$ ($\not\equiv 0$), where $\varphi_1 > 0$ is the first eigenfunction of A_p. But this is impossible in equation (2.14). □

In the case that $0 \notin \sigma(A_p)$, the spectrum of A_p, the relation (2.13) with (2.14) reduces to

$$\frac{\rho}{\lambda}\int_\Omega p\{1 + A_p^{-1}(p)\}\,dx = 0 \quad \text{with } v = \frac{\rho}{\lambda}A_p^{-1}(p)$$

because $S = \int_\Omega p\,dx$. Therefore, $T = d_h\Phi$ is invertible if and only if

$$I \equiv -\int_\Omega p\{1 + A_p^{-1}(p)\}\,dx \neq 0.$$

Further,

LEMMA 2. *We have $\partial S/\partial\lambda = -I/\lambda$ if $0 \notin \sigma(A_p)$.* □

PROOF. In that case, the section u of (P) is smooth with respect to λ. Actually, we get

$$v = \frac{\partial}{\partial\lambda}u = \frac{1}{\lambda}A_p^{-1}(p),$$

by differentiating (P) in λ. Therefore,

$$\frac{\partial S}{\partial\lambda} = \int_\Omega\{e^u + \lambda e^u v\}\,dx = \frac{1}{\lambda}\int_\Omega p\{1 + A_p^{-1}(p)\}\,dx. \quad \square$$

Under these preparations, we conclude that

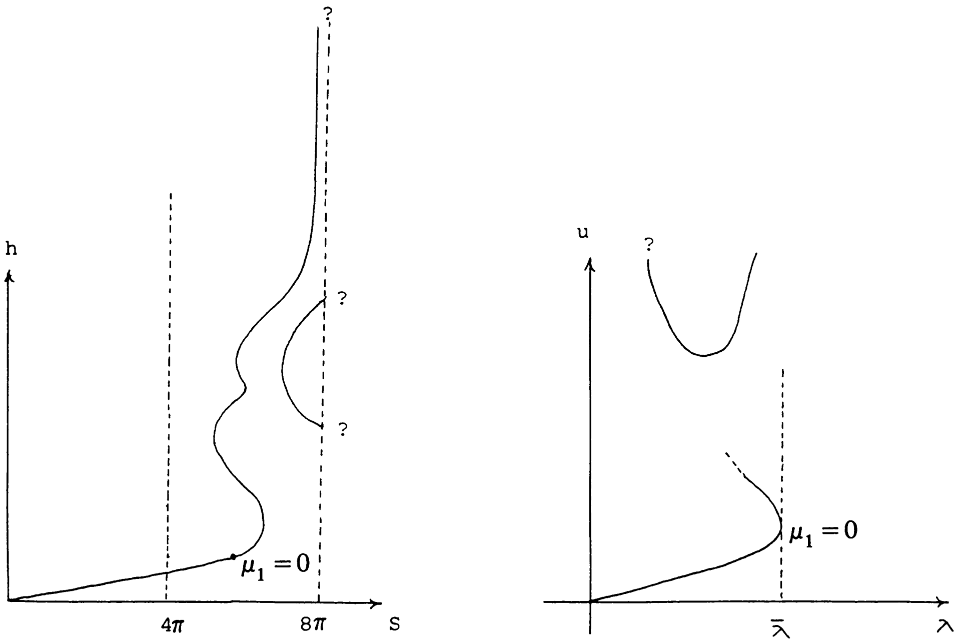
PROPOSITION 4. *In the case of $\Omega = D$, every solution $h = {}^T(u,\lambda)$ of (P) is parametrized by $S = \lambda\int_\Omega e^u\,dx \in (0, 8\pi)$. Let it be $h_0(S) = {}^T(u_0(S), \lambda_0(S))$. Then, $d_h\Phi(h_0(S), S)\colon \hat{X} \to \hat{Y}$ is invertible at each $S \in (0, 8\pi)$.* □

PROOF. According to the explicit formula (1.4), every solution $h = {}^T(u,\lambda)$ is reparametrized by $S \in (0,8\pi)$: $h = h_0(S) = {}^T(u_0(S), \lambda_0(S))$ $(0 < S < 8\pi)$.

The inverse mapping of $S \in (0, 8\pi) \mapsto \lambda_0(S) \in (0,2)$ is two-valued: $S = S_0^\pm(\lambda)$, where $S_0^\pm(\lambda) \to 4\pi$ as $\lambda \to 2$ and $S_0^+(\lambda) \to 8\pi$, $S_0^-(\lambda) \to 0$ as $\lambda \to 0$. Therefore, $\mu_1(p_0(S)) > 0$ for $0 < S < 4\pi$, $\mu_1(p_0(S)) = 0$ for $S = 4\pi$ and $\mu_1(p_0(S)) < 0$ for $4\pi < S < 8\pi$ from the local theory of Crandall and Rabinowitz [**4**], where $p_0(S) = \lambda_0(S)e^{u_0(S)}$. Hence $d_h\Phi(h_0(S), S)$ is invertible for $0 < S \leq 4\pi$ by Lemma 1. On the other hand, in the case $4\pi < S < 8\pi$ we have $0 \notin \sigma(A_p)$ by Corollary 1. Then, $\partial S_0^+(\lambda)/\partial\lambda \neq 0$ $(0 < \lambda < 2)$ is verified directly by (1.4), so that $d_h\Phi(h_0(S), S)$ $(4\pi < S < 8\pi)$ is invertible by Lemma 2.

3. Theorems and proofs. In what follows, we seek the solutions $(h, S) \in \hat{X}_+ \times \mathbf{R}$ of $\Phi(h, S) = 0$. There is a branch $\underline{\mathscr{S}}$ of zeros of Φ originating from $(h, S) = (0, 0)$, and corresponding to the branch of minimal solutions $\underline{\mathscr{C}}$ for (P) described in §1.

THEOREM 1. *Every zero point (h_0, S_0) of Φ generates a branch $\mathscr{S}_0$ of $\Phi(h, S) = 0$ in the S-h plane, whenever $S_0 < 8\pi$. Each end of $\mathscr{S}_0$ approaches eventually either the hyperplane $S = 8\pi$ or else $(0, 0)$. In the latter case, that is, when $\mathscr{S}_0$ is connected with $\underline{\mathscr{S}}$, the branch formed in this way bends at most once in the λ-u plane.* □



PROOF. Set $p_0 = \lambda_0 e^{u_0}$, where $h_0 = {}^T(u_0, \lambda_0)$. Then, $\mu_2(p_0) > 0$ holds by Corollary 1.

In the case of $\mu_1(p_0) \neq 0$, the implicit function theorem applies to problem (P) with respect to the parameter λ, and (h_0, S_0) generates a branch $\mathscr{S}_0$ of zeros of Φ. In the case $\mu_1(p_0) = 0$, on the other hand, Lemma 1 is available and we get the same conclusion.

Henceforth, we set $p = \lambda e^u$ for $h = {}^T(u, \lambda)$ where $\Phi(h, S) = 0$ holds with some S. We shall show the global behavior of $\mathscr{S}_0$.

Along one direction of that branch $\mathscr{S}_0$, suppose that the relation $S \leq S_1$ always holds with an $S_1 < 8\pi$. Then, we have $\mu_2(p) > 0$ along those zeros of Φ. We shall show that there eventually appears a point (h_1, S_1) in S_0 such that $\mu_1(p_1) > 0$, where $p_1 = \lambda_1 e^{u_1}$. Then, such an (h_1, S_1) lies on the minimal branch $\underline{\mathscr{S}}$, which originates from $(0, 0)$.

To this end, we first show that along that direction with $S \leq S_1$ $(< 8\pi)$, it is impossible for $\mu_1(p) < 0 < \mu_2(p)$ to keep holding. Suppose the contrary. Then, there is a branch $\mathscr{C}_0$ of the solutions of (P) in the λ-u plane corresponding to $\mathscr{S}_0$.

The implicit function theorem holds along the corresponding direction of $\mathscr{C}_0$ with respect to λ from the above assumption. On the other hand, we have an a priori estimate in Proposition 2, so that $\mathscr{C}_0$ continues up to either $\lambda \to +\infty$ or $\lambda \to 0$. However, the estimate $\lambda \leq \bar{\lambda}(\Omega)$ holds and $\lambda \to +\infty$ is impossible. Thus, $\mathscr{C}_0$ continues to $(0,0)$, because $u = 0$ is the unique section at $\lambda = 0$ of (P). However, $\mu_1(p) > 0$ holds near $(0,0)$ on $\mathscr{C}_0$, and hence this case does not occur.

Next we show that when $\mu_1(p_*) = 0$ occurs at some point $(h_*, S_*) \in \mathscr{S}_0$, then $\mu_1(p)$ changes sign near p_* on $\mathscr{S}_0$, where $p_* = \lambda_* e^{u_*}$ for $h_* = {}^T(u_*, \lambda_*)$. This fact, together with the above one, will imply the connectivity of $\mathscr{S}_0$ and $\underline{\mathscr{S}}$ for all cases.

To verify this fact, we recall the local theory of Crandall and Rabinowitz. Namely, in the case $\mu_1(p_*) = 0$, near $h_* = {}^T(u_*, \lambda_*)$ $\mathscr{C}_0$ is parametrized as $\{(\lambda(t), u(t)) \mid |t| < \varepsilon_0\}$ with

$$u(t) = u_* + t\varphi_{1^*} + o(t) \quad \text{and} \quad \lambda(t) = \lambda_* + ct^2 + o(t^2),$$

where $\varphi_{1_*} > 0$ denotes the first eigenfunction of A_{p_*} [**23**, Theorem 3.2]. Further, the computation of Theorem 4.8 of [**23**] shows that $\ddot{\lambda}(0) < 0$. Here we have

$$-\Delta u(t) = \lambda(t) e^{u(t)} \quad (\text{in } \Omega), \qquad u(t) = 0 \quad (\text{on } \partial\Omega).$$

Hence for $\dot{u}(t) = \partial u(t)/\partial t$ and $p(t) = \lambda(t)e^{u(t)}$ we obtain

$$-\Delta \dot{u}(t) = p(t)\dot{u}(t) + \dot{\lambda}(t)e^{u(t)} \quad (\text{in } \Omega), \qquad \dot{u}(t) = 0 \quad (\text{on } \partial\Omega)$$

so that

$$T(t) \equiv \int_\Omega |\nabla \dot{u}(t)|^2\, dx - \int_\Omega p(t)\dot{u}(t)^2\, dx = \int_\Omega \dot{\lambda}(t) e^{u(t)} \dot{u}(t)\, dx.$$

Because of $\ddot{\lambda}(0) < 0$, we have $\dot{\lambda}(t) \neq 0$ $(0 < |t| < \varepsilon_0)$ for $\varepsilon_0 > 0$ sufficiently small. This means that $\mu_1(p(t)) \neq 0$ $(0 < |t| < \varepsilon_0)$, because $\mu_1(p(t)) = 0$ for $t = t_0$ implies that $\dot{\lambda}(t_0) = 0$ by the local theory. Further, we have

$$T'(0) = \int_\Omega \ddot{\lambda}(0) e^{u_*} \dot{u}(0)\, dx = \ddot{\lambda}(0) \int_\Omega e^{u_*} \varphi_{1^*}\, dx < 0$$

with $T(0) = 0$ and hence $\mu_1(p(t)) < 0$ for $0 < t < \varepsilon_0$.

Now, we shall show that $\mu_1(p(t)) > 0$ for $-\varepsilon_0 < t < 0$.

In fact, we have shown that it is impossible for $\mu_1(p) < 0$ to keep holding along the direction of $\mathscr{C}_0$ in consideration. Therefore, in case $\mu_1(p(t)) < 0$ for $-\varepsilon_0 < t < 0$, we have to meet the next point $h_{**} = {}^T(u_{**}, \lambda_{**})$ on $\mathscr{C}_0$ such that $\mu_1(p_{**}) = 0$ for $p_{**} = \lambda_{**}e^{u_{**}}$. But this is impossible, because we must also have $\lambda'' < 0$ at h_{**} from the calculation of [**23**] mentioned above. Thus, we see that along the direction $\mathscr{C}_0$ in consideration, the parameter $t \in (-\varepsilon_0, \varepsilon_0)$ decreases from ε_0 to $-\varepsilon_0$ and that $\mu_1(p(t)) > 0$ holds for $-\varepsilon_0 < t < 0$.

In this way, we have shown that in the case that the relation $S \leq S_1$ $(< 8\pi)$ is preserved along one direction of $\mathscr{S}_0$, (h_0, S_0) connects with $(0,0)$, and furthermore the corresponding branch $\mathscr{C}_0$ in the λ-u plane bends at most once. □

Next, we suppose that Ω is star-shaped with respect to the origin and put $B \equiv \int_{\partial\Omega} ds/(n \cdot x)$, where n denotes the outer unit normal vector on $\partial\Omega$. Then, we have $B \geq 2\pi$, where the equality holds when Ω is a disc.

If $h = {}^T(u, \lambda)$ solves (P), the estimate

$$(S - 2B)^2/B \leq 4B - 4\lambda|\Omega|$$

holds by Rellich's identity, where $S = \lambda \int_\Omega e^u\,dx$ (Bandle [**3**, p. 202]). In particular, $B \le 4\pi$ and $S \ge 8\pi$ imply that

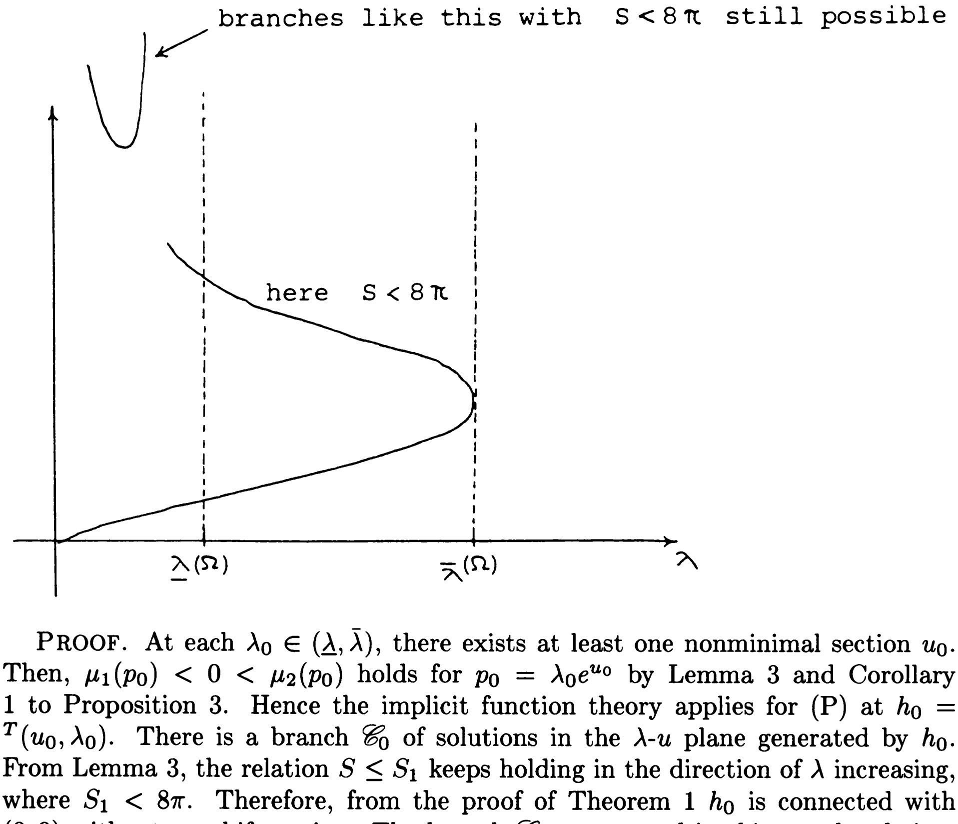
$$(8\pi - 2B)^2/B \le (S - 2B)^2/B \le 4B - 4\lambda|\Omega|,$$

and hence $\lambda \le 8\pi(B - 2\pi)/|\Omega|B$. In other words, $S < 8\pi$ holds when $\lambda > \underline{\lambda}(\Omega) \equiv 8\pi(B - 2\pi)/|\Omega|B$ and $B \le 4\pi$. More precisely,

LEMMA 3. *In the case of $B \le 4\pi$, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\lambda \ge \underline{\lambda} + \varepsilon$ implies $S \le 8\pi - \delta$.* □

Now, the next theorem follows from the previous one.

THEOREM 2. *If Ω is star-shaped with respect to the origin, $B = \int_{\partial\Omega} ds/(n\cdot x) \le 4\pi$ and $\underline{\lambda}(\Omega) < \bar{\lambda}(\Omega)$, then for each λ in $\underline{\lambda} < \lambda < \bar{\lambda}$, the problem* (P) *has exactly two sections, that is, the minimal section and the nonminimal one. In the λ-u plane, these are connected to each other.* □



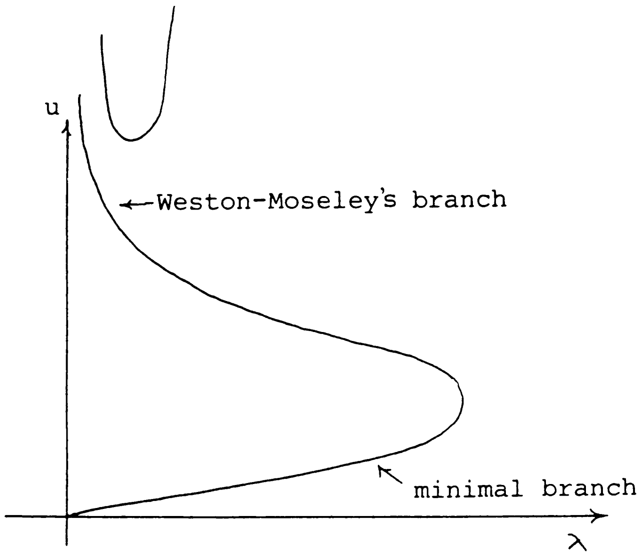
PROOF. At each $\lambda_0 \in (\underline{\lambda}, \bar{\lambda})$, there exists at least one nonminimal section u_0. Then, $\mu_1(p_0) < 0 < \mu_2(p_0)$ holds for $p_0 = \lambda_0 e^{u_0}$ by Lemma 3 and Corollary 1 to Proposition 3. Hence the implicit function theory applies for (P) at $h_0 = {}^T(u_0, \lambda_0)$. There is a branch $\mathscr{C}_0$ of solutions in the λ-u plane generated by h_0. From Lemma 3, the relation $S \le S_1$ keeps holding in the direction of λ increasing, where $S_1 < 8\pi$. Therefore, from the proof of Theorem 1 h_0 is connected with $(0,0)$ without any bifurcation. The branch $\mathscr{C}$ constructed in this way bends just once. Further, any nonminimal solution $\tilde{h} = {}^T(\tilde{u}, \tilde{\lambda})$ with $\tilde{\lambda} \in (\underline{\lambda}, \bar{\lambda})$ generates a branch $\tilde{\mathscr{C}}$, which is connected with $\mathscr{C}$. Since $\mathscr{C}$ has no bifurcation, we conclude that $\tilde{h} \in \mathscr{C}$. □

Finally, we shall show our main result, that is, the branch of Weston-Moseley's large solutions connects with that of minimal solutions when Ω is close to a disc.

To this end, let $\omega \subset \mathbf{R}^2$ be a simply connected domain with smooth boundary $\partial\omega$, and let $g_1\colon D \to \omega$ be a Riemann mapping such that $g_1''(0) = 0$. Actually, such a g_1 exists as we have shown in §2.1. For sufficiently small $|\varepsilon|$, let $g_{N,\varepsilon} = g_{N,\varepsilon}(\varsigma) = \varsigma + \varepsilon g_1(\varsigma)\colon D \to \Omega_\varepsilon$, where $\Omega_\varepsilon = g_{N,\varepsilon}(D)$. Then, $g_{N,\varepsilon}$ becomes a Riemann mapping satisfying $g''_{N,\varepsilon}(0) = 0$. In fact, univalentness follows from Darboux's theorem.

If $|\varepsilon|$ is small, $\alpha_\varepsilon = |g'''_{N,\varepsilon}(0)/g'_{N,\varepsilon}(0)| < 2$ holds, so that the branch of Weston-Moseley's large solutions for (P) can be constructed in Ω_ε, which is denoted by $\mathscr{C}_\varepsilon^* = \{(\lambda, u^*_{\lambda,\varepsilon})\}$. On the other hand, there exists the branch of minimal solutions in Ω_ε denoted by $\underline{\mathscr{C}}_\varepsilon$. Then,

THEOREM 3. *If $|\varepsilon|$ is sufficiently small, $\mathscr{C}_\varepsilon^*$ connects with $\underline{\mathscr{C}}_\varepsilon$. Further, the branch $\mathscr{C}_\varepsilon$ constructed in this way bends just once in the λ-u plane. Namely, we can parametrize $\mathscr{C}_\varepsilon = \{(\lambda_t, u_t) | 0 \le t < 3\}$ as $(u_0, \lambda_0) = (0,0)$ and λ_t increases in $t \in (0, \bar{t})$ and decreases in $t \in (\bar{t}, 3)$ with some $\bar{t} \in (0,3)$. Furthermore, here we have $\lambda_{\bar{t}} = \bar{\lambda}(\Omega_\varepsilon)$.*



PROOF. According to the formulation in §2.3, we can transform problem (P) in Ω_ε to finding zeros of the mapping $\Phi = \Phi_\varepsilon$ defined below. Namely, $X_\varepsilon = C_0^{2+\alpha}(\bar{\Omega}_\varepsilon)$, $Y_\varepsilon = C^\alpha(\bar{\Omega}_\varepsilon)$, $\hat{X}_\varepsilon = \underset{\mathbf{R}}{\overset{X_\varepsilon}{\times}}$, $\hat{X}_{\varepsilon'+} = \underset{\mathbf{R}+}{\overset{X_\varepsilon}{\times}}$, $\hat{Y}_\varepsilon = \underset{\mathbf{R}}{\overset{Y_\varepsilon}{\times}}$, and $\Phi_\varepsilon = \Phi_\varepsilon(h,S)\colon \hat{X}_{\varepsilon'+} \times \mathbf{R} \to \hat{Y}_\varepsilon$, where

$$\Phi_\varepsilon(h,S) = \begin{pmatrix} \Delta u + \lambda e^u \\ \int_{\Omega_\varepsilon} e^u\,dx - \frac{S}{\lambda} \end{pmatrix} \quad \text{for } h = {}^T(u,\lambda).$$

Corresponding to the minimal branch $\underline{\mathscr{C}}_\varepsilon$, there is a branch $\underline{\mathscr{S}}_\varepsilon$ of zeros of Φ_ε in the h-S plane, originating from $(h,S) = (0,0)$. By virtue of Theorem 1, $\underline{\mathscr{S}}_\varepsilon$ approaches eventually the hyperplane $S = 8\pi$. Let $\tilde{\mathscr{S}}_\varepsilon$ be the branch generated by $\underline{\mathscr{S}}_\varepsilon$ in this way.

On the other hand, along the branch $\mathscr{C}_\varepsilon^*$ of large solutions, the quantity $S = \lambda \int_\Omega e^{u^*_{\lambda,\varepsilon}}\,dx$ tends to 8π from below as $\lambda \downarrow 0$ by Proposition 1. Therefore, λ is parametrized by S and hence $\mathscr{C}_\varepsilon^*$ can be reparametrized as $\mathscr{C}_\varepsilon^* = \{(\lambda(S), u_\varepsilon^*(S)) | S_0 < S < 8\pi\}$ with an $S_0 \in (0, 8\pi)$. Further, S_0 can be taken to be independent of ε in $|\varepsilon| < \varepsilon_1$, where $\varepsilon_1 > 0$ is a small constant, by virtue of (2.9) and (2.10). Actually, (2.9) holds uniformly in ε. Henceforth, we put $h_\varepsilon^*(S) = {}^T(u_\varepsilon^*(S), S)$ $(S_0 < S < 8\pi)$: $\Phi_\varepsilon(h_\varepsilon^*(S), S) = 0$ $(|\varepsilon| < \varepsilon_0,\ S_0 < S < 8\pi)$.

From the Riemann mapping $g_{N,\varepsilon}\colon \Omega_0 \to \Omega_\varepsilon$, the problem (P) on Ω_ε is pulled back to that on $D = \Omega_0$:

$$-\Delta U = \lambda |g'_{N,\varepsilon}|^2 e^U \quad (\text{in } D) \tag{3.1}$$

with

$$U = 0 \quad (\text{on } \partial D). \tag{3.2}$$

Then, Φ_ε is transformed into the operator $F_\varepsilon\colon \hat{X}_{0,+} \times \mathbf{R} \to \hat{Y}_0$ as

$$F_\varepsilon(H, S) = \begin{pmatrix} \Delta U + \lambda |g'_{N,\varepsilon}|^2 e^U \\ \int_{\Omega_0} |g'_{N,\varepsilon}|^2 e^U \, dx - \frac{S}{\lambda} \end{pmatrix},$$

where $H = {}^T(U, \lambda)$.

For the large solution $u^* = u_{\lambda,\varepsilon}^*$, we set $U_{\lambda,\varepsilon}^* = u_{\lambda,\varepsilon}^* \circ g_N$, and $H_\varepsilon^* = {}^T(U_{\lambda,\varepsilon}^*, \lambda)$. Then, H_ε^* is parametrized by $S \in (S_0, 8\pi)$ like h_ε^*, and the relation

$$F_\varepsilon(H_\varepsilon^*(S), S) = 0 \tag{3.3}$$

follows for $|\varepsilon| < \varepsilon_0$ and $S_0 < S < 8\pi$. Furthermore,

$$||U_{\lambda,\varepsilon}^*||_{C^0(\bar{D})} \le -2\log(1 - S/8\pi) \tag{3.4}$$

holds by Proposition 2, so that $\{H_\varepsilon^*(S_1) | \, |\varepsilon| \le \varepsilon_0/2\}$ is compact in $\hat{X}_0$ for each fixed $S_1 \in (S_0, 8\pi)$ by virtue of the elliptic estimate.

Taking a suitable sequence $\{\varepsilon_j\}$ with $\varepsilon_j \to 0$, $H_{\varepsilon_j}^*(S_1)$ converges in $\hat{X}_0$. Then, the limit $\tilde{H}_0^*(S_1)$ solves $\Phi_0(\tilde{H}_0^*(S_1), S_1) = F_0(\tilde{H}_0^*(S_1), S_1) = 0$. However, as we have shown in Proposition 4, the zero of $\Phi_0(\cdot, s_1)$ is unique, that is, $h_0(S_1)$. Hence

$$H_\varepsilon^*(S_1) \to h_0(S_1) \quad \text{as } \varepsilon \to 0 \text{ in } \hat{X}_0. \tag{3.5}$$

On the other hand, the branch $\tilde{\mathscr{S}}_\varepsilon$ generated by the minimal one has at least one section at $S = S_1$, which is denoted by $\underline{h}_\varepsilon(S_1) \in \hat{X}_\varepsilon$. Similarly, $\underline{h}_\varepsilon(S_1)$ is transformed into an $\underline{H}_\varepsilon(S_1) \in \hat{X}_0$ through $g_{N,\varepsilon}\colon \Omega_0 \to \Omega_\varepsilon$ with the relation $F_\varepsilon(\underline{H}_\varepsilon(S_1), S_1) = 0$. In the same way, we have

$$\underline{H}_\varepsilon(S_1) \to h_0(S_1) \quad \text{as } \varepsilon \to 0 \text{ in } \hat{X}_0. \tag{3.6}$$

Now, Proposition 4 indicates that the operator $T_0 = d_H F_0(h_0(S_1), S_1);\ \hat{X}_0 \to \hat{Y}_0$ is invertible. Therefore, the same is true for the operator $T_\varepsilon = d_H F_\varepsilon(H_\varepsilon^*(S_1), S_1)$: $\hat{X}_0 \to \hat{Y}_0$, provided that $|\varepsilon|$ is small. In particular, the equation

$$F_\varepsilon(H, S_1) = 0 \tag{3.7}$$

has the local uniqueness property around the solution $H = H_\varepsilon^*(S_1)$ uniformly in ε. Namely, there exist some $\varepsilon_1 > 0$ and $\kappa > 0$ such that $|\varepsilon| \le \varepsilon_1$, $F_\varepsilon(H, S_1) = 0$ and $||H - H_\varepsilon^*(S_1)||_{X_0} < \kappa$ imply $H = H_\varepsilon^*(S_1)$. Therefore, by virtue of (3.5) and (3.6), we get $H_\varepsilon^*(S_1) = \underline{H}_\varepsilon(S_1)$ to conclude that $\mathscr{S}_\varepsilon^*$ and $\underline{\mathscr{S}}_\varepsilon$, and hence $\mathscr{C}_\varepsilon^*$ and $\underline{\mathscr{C}}_\varepsilon$ connect to each other when $|\varepsilon|$ is sufficiently small.

The latter part of the theorem follows from Theorem 1. $\square$

Appendix I.

PROOF OF (2.9). Let u_0 be the fifth-order asymptotic solution. We first will show that (2.9) is reduced to

$$S_0 \equiv \lambda \int_\Omega e^{u_0}\, dx = 8\pi + C\lambda + o(\lambda) \quad \text{as } \lambda \downarrow 0. \tag{I.1}$$

In fact, then we get

$$\begin{aligned} |S - S_0| &\leq \lambda \int_\Omega e^{u_0}\, dx \{ e^{\|u-u_0\|_{C^0(\bar{D})}} - 1\} \\ &= S_0 \{ e^{\|u-u_0\|_{C^0(\bar{D})}} - 1\} \leq C\lambda^2 \end{aligned}$$

by (2.8).

To show (I.1), we put $U = u_0 \circ g_N$. Then U satisfies

$$-\Delta u = \lambda |g_N'|^2 e^U \quad (\text{in } D),$$

so that

$$S_0 = \lambda \int_\Omega e^{u_0}\, dx = \lambda \int_D e^U |g_N'|^2\, dx = -\int_D \Delta U\, dx = -\int_{\partial D} \frac{\partial U}{\partial r}\, ds, \tag{I.2}$$

where $r = |x|$.

The asymptotic solution U is given as

$$e^{-U/2} = \frac{\{|\varsigma|^2 + (\lambda/8)|A(\varsigma)|^2\}}{|G(\varsigma)|^2} \qquad (\varsigma \in D), \tag{I.3}$$

where $G(\varsigma) = G(\varsigma, \lambda) = 1 + \lambda G_1(\varsigma) + \cdots + \lambda^{n-1} G_{n-1}(\varsigma)$ $(n = 5)$ and

$$A(\varsigma) = A(\varsigma, \lambda) = \varsigma \int^\varsigma G(\hat{\varsigma}, \lambda)^2 \frac{g_N^1(\hat{\varsigma})}{\hat{\varsigma}^2}\, d\hat{\varsigma},$$

which are described more precisely later [**16**]. Hence

$$\begin{aligned} -\frac{1}{2}\frac{\partial U}{\partial r} e^{-U/2} = &\left\{ 2r + \frac{\lambda}{8}\frac{\partial}{\partial r}|A(\varsigma,\lambda)|^2 \right\} \Big/ |G(\varsigma,\lambda)|^2 \\ &- \left\{ r^2 + \frac{\lambda}{8}|A(\varsigma,\lambda)|^2 \right\} \frac{\partial}{\partial r}|G(\varsigma,\lambda)|^2 \Big/ |G(\varsigma,\lambda)|^4, \end{aligned}$$

so that

$$-\frac{1}{2}\frac{\partial U}{\partial r}\bigg|_{r=1,\lambda=0} = 2$$

by $G(\varsigma, 0) = 1$ and $U|_{\partial D} = O(\lambda^n)$. Therefore, we get

$$S_0 = -\int_{\partial D} \frac{\partial U}{\partial r}\, ds = 8\pi + O(\lambda) \quad \text{as } \lambda \downarrow 0. \tag{I.1$'$}$$

Next we have

$$\begin{aligned} &-\frac{1}{2}\frac{\partial}{\partial\lambda}\frac{\partial U}{\partial r} e^{-U/2} + \frac{1}{4}\frac{\partial U}{\partial r}\frac{\partial U}{\partial \lambda} e^{-U/2} \\ &\quad = \frac{\partial}{\partial\lambda}\left\{ \left(2r + \frac{\lambda}{8}\frac{\partial}{\partial r}|A(\varsigma,\lambda)|^2 \right) \Big/ |G(\varsigma,\lambda)|^2 \right\} \\ &\quad\quad - \frac{\partial}{\partial\lambda}\left\{ \left(r^2 + \frac{\lambda}{8}|A(\varsigma,\lambda)|^2 \right) \frac{\partial}{\partial r}|G(\varsigma,\lambda)|^2 \Big/ |G(\varsigma,\lambda)|^4 \right\} \\ &\quad = \mathrm{I} - \mathrm{II} \end{aligned}$$

with

$$\mathrm{I} = \left\{ \frac{1}{8}\frac{\partial}{\partial r}|A(\varsigma,\lambda)|^2 + \frac{\lambda}{8}\frac{\partial}{\partial \lambda}\frac{\partial}{\partial r}|A(\varsigma,\lambda)|^2 \right\} \Big/ |G(\varsigma,\lambda)|^2 - \left\{ 2r + \frac{\lambda}{8}\frac{\partial}{\partial r}|A(\varsigma,\lambda)|^2 \right\} \frac{\partial}{\partial \lambda}|G(\varsigma,\lambda)|^2 \Big/ |G(\varsigma,\lambda)|^4$$

and

$$\begin{aligned} \mathrm{II} = \Bigg\{ &\left(\frac{1}{8}|A(\varsigma,\lambda)|^2 + \frac{\lambda}{8}\frac{\partial}{\partial \lambda}|A(\varsigma,\lambda)|^2 \right) \frac{\partial}{\partial r}|G(\varsigma,\lambda)|^2 \\ &+ \left(r^2 + \frac{\lambda}{8}|A(\varsigma,\lambda)|^2 \right) \frac{\partial}{\partial \lambda}\frac{\partial}{\partial r}|G(\varsigma,\lambda)|^2 \Bigg\} \Big/ |G(\varsigma,\lambda)|^4 \\ &- 2\left(r^2 + \frac{\lambda}{8}|A(\varsigma,\lambda)|^2 \right) \frac{\partial}{\partial r}|G(\varsigma,\lambda)|^2 \frac{\partial}{\partial \lambda}|G(\varsigma,\lambda)|^2 \Big/ |G(\varsigma,\lambda)|^6. \end{aligned}$$

Therefore, we have

$$\mathrm{I}|_{\lambda=0} = \frac{1}{8}\frac{\partial}{\partial r}|A_0(\varsigma)|^2 - 4r \operatorname{Re} G_1(\varsigma)$$

and

$$\mathrm{II}|_{\lambda=0} = 2r^2 \frac{\partial}{\partial r} \operatorname{Re} G_1(\varsigma),$$

where $A_0(\varsigma) = A(\varsigma, 0)$. Hence

$$\begin{aligned} &-\frac{1}{2}\frac{\partial}{\partial \lambda}\frac{\partial U}{\partial r}\bigg|_{\lambda=0,r=1} \\ &= \left\{ \frac{1}{8}\frac{\partial}{\partial r}|A_0(\varsigma)|^2 - 4 \operatorname{Re} G_1(\varsigma) - 2\frac{\partial}{\partial r} \operatorname{Re} G_1(\varsigma) \right\}\bigg|_{|\varsigma|=1}. \end{aligned} \tag{I.4}$$

We recall the relations in [16], that is,

$$2 \operatorname{Re} G_1(\varsigma) = \tfrac{1}{8}\{|C_0|^2 + 2\operatorname{Re}(-g_N'(0)C_0\varsigma + \bar{C}_0 I_0(\varsigma)) + |-g_N'(0) + \varsigma I_0(\varsigma)|^2\}$$

and

$$A_0(\varsigma) = -g_N'(0) + \varsigma I_0(\varsigma) + C_0 \varsigma,$$

where $C_0 \in \mathbf{C}$ is a constant and

$$I_0(\varsigma) = \int_0^{\varsigma} (g_N'(\hat{\varsigma}) - g_N'(0)) \frac{d\hat{\varsigma}}{\hat{\varsigma}^2}.$$

Here, g_N is normalized as $g_N'(0) > 0$ so that

$$2 \operatorname{Re} G_1(\varsigma) = \tfrac{1}{8}|A_0(\varsigma)|^2 \quad (\text{on } |\varsigma| = 1). \tag{I.5}$$

Furthermore, $G_1 = G_1(\varsigma)$ is holomorphic in D and hence

$$-\int_{\partial D} \frac{\partial}{\partial r} \operatorname{Re} G_1 \, ds = -\int_D \Delta(\operatorname{Re} G_1)\, dx = 0. \tag{I.6}$$

Therefore, the relation (I.1) holds with

$$\begin{aligned} C &= -\int_{\partial D} \frac{\partial}{\partial \lambda}\left(\frac{\partial U}{\partial r}\right)\bigg|_{r=1} ds \\ &= \int_{\partial D} \left\{ \frac{1}{4}\frac{\partial}{\partial r}|A_0(\varsigma)|^2 - \frac{1}{2}|A_0(\varsigma)|^2 \right\} ds. \end{aligned} \tag{I.7}$$

Setting $A_0(\varsigma) = \sum_{n=0}^{\infty} b_n \varsigma^n$, we have

$$\int_{\partial D} |A_0(\varsigma)|^2 \, ds = \sum_{n,m=0}^{\infty} \int_0^{2\pi} b_n \bar{b}_m e^{i(n-m)\theta} \, d\theta = 2\pi \sum_{n=0}^{\infty} |b_n|^2.$$

Similarly,

$$\int_{\partial D} \frac{\partial}{\partial r} |A_0(\varsigma)|^2 \, ds = \sum_{\substack{n,m=0 \\ n+m \geq 1}}^{\infty} (n+m) b_n \bar{b}_m \int_0^{2\pi} e^{i(n-m)} \, d\theta = 4\pi \sum_{n=1}^{\infty} n |b_n|^2,$$

so that

$$C = \left\{ -|b_0|^2 + \sum_{n=2}^{\infty} (n-1)|b_n|^2 \right\} \pi. \tag{I.8}$$

By virtue of $g_N(\varsigma) = \sum_{n=0}^{\infty} a_n \varsigma^n$ with $a_2 = 0$, we have

$$\begin{aligned} A_0(\varsigma) &= -g_N'(0) + C_0 \varsigma + \varsigma I_0(\varsigma) \\ &= -g_N'(0) + C_0 \varsigma + \sum_{n=2}^{\infty} a_{n+1} \frac{n+1}{n-1} \varsigma^n. \end{aligned}$$

Hence $b_0 = -g_N'(0) = -a_1$ and $b_n = a_{n+1}(n+1)/(n-1)$ $(n \geq 2)$. Thus, (2.10) follows. □

Appendix II.

PROOF OF PROPOSITION 1. Taking some constant ξ in $0 < \xi < 1$, we put $g_\xi(\varsigma) = (g_N(\varsigma) - g_N(0))/\xi g_N'(0)$ and $f_\xi(\varsigma) = \varsigma g_\xi'(\varsigma) = \sum_{n=0}^{\infty} d_n \varsigma^n$. Since g_n is univalent in D, so is also g_ξ. Then we have

$$d_n = \frac{1}{n!} f_\xi^{(n)}(0) = n \frac{1}{n!} g_\xi^{(n)}(0) = \frac{n}{\xi} \cdot \frac{a_n}{a_1}.$$

In particular, $d_0 = d_2 = 0$ and $d_1 = 1/\xi$. The relation $C < 0$ follows from

$$\sum_{n=1}^{\infty} \frac{1}{n} |d_{n+2}|^2 \quad \left(= \frac{1}{\xi^2} \sum_{n=3}^{\infty} \frac{n^2}{n-2} \left| \frac{a_n}{a_1} \right|^2 \right) \leq 1. \tag{II.1}$$

We consider the function

$$w_\xi(\varsigma) = \frac{1}{\varsigma} + \sum_{n=1}^{\infty} c_n \varsigma^n,$$

which is holomorphic in $0 < |\varsigma| < 1$, where $c_n = -d_{n+2}/n$. When w_ξ is univalent, the desired inequality (II.1) follows from the area theorem [**18**, p. 210];

$$\sum_{n=1}^{\infty} n |c_n|^2 = \sum_{n=1}^{\infty} \frac{1}{n} |d_{n+2}|^2 \leq 1.$$

The image Γ_r of $c_r = \{|z| = r\}$ $(0 < r < 1)$ by w_ξ is a closed curve. The univalentness of w_ξ follows if Γ_r is a Jordan curve and the winding number of the mapping $\varsigma \in c_r \mapsto w_\xi(\varsigma) \in \Gamma_r$ is -1 for each r close to 1.

In fact, let $\bar{\mathbf{C}}$ be the Riemann sphere $\mathbf{C} \cup \{\infty\}$ and $\mathscr{S}\colon \mathbf{C} \to \bar{\mathbf{C}}$ be the canonical injection. The pole $\varsigma = 0$ of w_ξ is first order, and hence w_ξ extends conformally as

$\mathscr{S} \circ w_\xi \colon D \to \bar{\mathbf{C}}$. From the above assumption, we can take a mapping $\mathscr{T} \colon \bar{\mathbf{C}} \to \bar{\mathbf{C}}$, which is nothing but a rotation of the Riemann sphere, so that the image of $\mathscr{T} \circ \mathscr{S} \circ w_\xi \colon D_r = \{|z| < r\} \to \bar{\mathbf{C}}$ does not contain ∞. Therefore, the mapping $\mathscr{S}_{-1} \circ \mathscr{T} \circ \mathscr{S} \circ w_\xi$ is holomorphic in D_r with the image Ω_r surrounded by a Jordan curve γ_r, where $\mathscr{S}_{-1} \colon \bar{\mathbf{C}} \backslash \{\infty\} \to \mathbf{C}$ denote the canonical projection.

This time, the winding number of

$$\varsigma \in c_r \mapsto \mathscr{S}_{-1} \circ \mathscr{T} \circ \mathscr{S} \circ w_\xi(\varsigma) \in \gamma_r$$

is $+1$ and $\mathscr{S}_{-1} \circ \mathscr{T} \circ \mathscr{S} \circ w_\xi$ is univalent in D_r from Darboux's theorem. Therefore, the same is true for w_ξ in $0 < |\varsigma| < 1$, because r can be taken arbitrarily close to 1.

Now, the relation

$$w_\xi'(\varsigma) = -\frac{1}{\varsigma^3} f_\xi(\varsigma) + \left(\frac{1}{\xi} - 1\right)\frac{1}{\varsigma^2} = -\frac{1}{\varsigma^2} g_\xi'(\varsigma) + \left(\frac{1}{\xi} - 1\right)\frac{1}{\varsigma^2} \tag{II.2}$$

is derived from $d_0 = d_2 = 1$ and $d_1 = 1/\xi$. In fact, we have

$$(w_\xi(\varsigma) - 1/\varsigma)' = -\varsigma^{-3}(f_\xi(\rho) - d_1\varsigma).$$

Therefore, for $\varsigma = re^{i\theta}$ $(0 \le \theta \le 2\pi)$ we have

$$\frac{\partial}{\partial\theta} w_\xi(re^{i\theta}) = i\varsigma w_\xi'(\varsigma) = -\frac{1}{\varsigma^2}(i\varsigma h_\xi'(\varsigma)) = -\frac{1}{\varsigma^2}\frac{\partial}{\partial\theta} h_\xi(re^{i\theta}),$$

where $h_\xi(\varsigma) = g_\xi(\varsigma) + (1 - 1/\xi)\varsigma$. Hence we get the relation

$$S_{r,\xi}(\theta) = e^{-2i\theta} T_{r,\xi}(\theta), \tag{II.3}$$

where

$$S_{r,\xi}(\theta) = \frac{\partial}{\partial\theta} w_\xi(re^{i\theta}) \Big/ \left|\frac{\partial}{\partial\theta} w_\xi(re^{i\theta})\right| \in S^1$$

and

$$T_{r,\xi}(\theta) = \frac{\partial}{\partial\theta} h_\xi(re^{i\theta}) \Big/ \left|\frac{\partial}{\partial\theta} h_\xi(re^{i\theta})\right| \in S^1.$$

The holomorphic function $g_\xi = g_\xi(\varsigma)$ is univalent for each $\xi > 0$, so that the winding number of $\varsigma = re^{i\theta} \in c_r \mapsto \tilde{T}_{r,\xi}(\theta) \in S^1$ is equal to $+1$, where

$$\tilde{T}_{r,\xi}(\theta) = \frac{\partial}{\partial\theta} g_\xi(re^{i\theta}) \Big/ \left|\frac{\partial}{\partial\theta} g_\xi(re^{i\theta})\right|.$$

Therefore, that of $\varsigma = re^{i\theta} \in c_r \mapsto T_{r,\xi}(\theta) \in S^1$ is also $+1$ whenever ξ in $0 < \xi < 1$ is close to 1. Consequently, the winding number of $\varsigma = re^{i\theta} \in c_r \mapsto S_{r,\xi}(\theta) \in S^1$ is equal to -1 by (II.3) when w_ξ is one-to-one on c_r. In this way, we have shown that (II.1) holds if w_ξ is one-to-one on $c_r = \{|z| = r\}$ when ξ and r in $(0,1)$ are close to 1.

A simple sufficient condition for that is

$$\frac{\partial}{\partial\theta}(\operatorname{Arg} S_{r,\xi}(\theta)) < 0 \qquad (0 \le \theta < 2\pi),$$

namely,

$$\frac{\partial}{\partial\theta}(\operatorname{Arg} T_{r,\xi}(\theta)) < 2 \qquad (0 \le \theta < 2\pi). \tag{II.4}$$

When ξ and r in $(0,1)$ are close to 1, (II.4) is implied by

$$\frac{\partial}{\partial\theta}(\operatorname{Arg} T(\theta)) < 2 \qquad (0 \le \theta < 2\pi), \tag{II.5}$$

where $T(\theta) = T_{1,1}(\theta) = g'_N(e^{i\theta})/|g'_N(e^{i\theta})| \in S^1$.

The unit tangent vector e_1 of $\partial\Omega$ at $g_N(e^{i\theta})$ is nothing but $T(\theta)$, and hence

$$e_1(l) = \begin{pmatrix} \cos t(\theta) \\ \sin t(\theta) \end{pmatrix},$$

where $t(\theta) = \operatorname{Arg} T(\theta)$ and $l = \int_0^\theta |g'_N(e^{i\omega})|\, d\omega$ represents the length parameter along $\partial\Omega$. Therefore, the inner unit normal vector on $\partial\Omega$ becomes

$$e_2(l) = \begin{pmatrix} \cos(t(\theta)+\pi/2) \\ \sin(t(\theta)+\pi/2) \end{pmatrix} = \begin{pmatrix} -\sin t(\theta) \\ \cos t(\theta) \end{pmatrix}.$$

Hence

$$e'_1(l) = \begin{pmatrix} -\sin t(\theta) \\ \cos t(\theta) \end{pmatrix} t'(\theta)\, \frac{d\theta}{dl} \qquad (= \kappa e_2(l)),$$

so that $t'(\theta) = \kappa |g'_N|$. In other words, the condition $\kappa|g'_N| < 2$ (on ∂D) implies $C < 0$. □

Appendix III.

PROOF OF PROPOSITION 2. Let $h = {}^T(u,\lambda)$ solve (P) and $S = \lambda \int_\Omega e^u\, dx$. For $t > 0$, set $\Omega_t = \{u > t\}$ and $\Gamma_t = \{u = t\}$. Then, by Green's formula we have

$$D(t) \equiv \int_{\Omega_t} \lambda e^u\, dx = -\int_{\Omega_t} \Delta u\, dx = -\int_{\Gamma_t} \frac{\partial u}{\partial n}\, ds = \int_{\Gamma_t} |\nabla u|\, ds. \tag{III.1}$$

On the other hand, from the co-area formula [**3**, p. 53] follows

$$D(t) = \int_t^\infty dr \int_{\Gamma_r} \lambda e^u \frac{1}{|\nabla u|}\, ds = \lambda \int_t^\infty e^r\, dr \int_{\Gamma_r} \frac{ds}{|\nabla u|},$$

and hence

$$\int_{\Gamma_t} \frac{ds}{|\nabla u|} = -\frac{1}{\lambda} D'(t) e^{-t}. \tag{III.2}$$

From these identities we obtain

$$-\frac{1}{\lambda} D'(t) D(t) e^{-t} \ge \left(\int_{\Gamma_t} ds \right)^2 = |\Gamma_t|^2. \tag{III.3}$$

Next, we have

$$\begin{aligned} |\Omega_t| &= \int_{\Omega_t} 1 \cdot dx = \int_t^\infty dr \int_{\Gamma_r} \frac{ds}{|\nabla u|} = -\frac{1}{\lambda} \int_t^\infty D'(r) e^{-r}\, dr \\ &= \frac{1}{\lambda} D(t) e^{-t} - \frac{1}{\lambda} \int_t^\infty D(r) e^{-r}\, dr. \end{aligned} \tag{III.4}$$

Combining (III.3), (III.4) with the isoperimetric inequality

$$|\Omega_t| \le |\Gamma_t|^2 / 4\pi,$$

we get

$$D(t) - \int_t^\infty e^{(t-s)} D(s)\, ds \le -\frac{1}{4\pi} D(t) D'(t) \equiv g(t) \qquad (\ge 0).$$

Let $H(t) = \int_t^\infty e^{(t-s)} D(s)\, ds$. Then, $-H'(t) = D(t) - H(t) \le g(t)$ so that $H(t) \le \int_t^\infty g(s)\, ds$. But $D(t) - H(t) \le g(t)$ or

$$D(t) \le g(t) + \int_t^\infty g(s)\, ds = -\frac{1}{4\pi} D(t) D'(t) + \frac{1}{8\pi} D(t)^2.$$

Therefore, $8\pi - D(t) \le -2D'(t)$ or

$$4\pi e^{-t/2} \le -(e^{-t/2} D(t))'. \tag{III.5}$$

Let $t_0 = ||u||_{C^0(\bar{\Omega})}$. Then,

$$\int_0^{t_0} 4\pi e^{-t/2}\, dt = 8\pi(1 - e^{-t_0/2}) \le -[e^{-t/2} D(t)]_{t=0}^{t=t_0} = S,$$

because $D(0) = S$ and $D(t(0)) = 0$. Hence we obtain

$$t_0 = ||u||_{C^0(\bar{\Omega})} \le -2\log(1 - S/8\pi).$$

ACKNOWLEDGMENTS. The authors thank heartily the referee for valuable advice in revising the manuscript. Thanks are also due to Professor T. Ochiai, for informing the first author of the paper by H. Wente and attracting him to Weston-Moseley's theory, to Professor J. L. Moseley for kindly sending us the unpublished paper [**15**], and to Professor H. Fujita for his interest in our work.

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